

Quantum Stochastic Heating of a Trapped Ion

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Abstract: The resonant heating of a harmonically trapped ion by a standing-wave light field is described as a quantum stochastic process combining a coherent Schrödinger evolution with Bohr-Einstein quantum jumps. Quantum and semi-quantum treatments are compared.
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1. Introduction

Trapped ions have featured prominently in the study of elementary quantum systems over a period of many years. Some of the earliest experiments demonstrated the phenomenon of electron shelving or quantum jumps [1-3]. Subsequently, with improved methods of laser cooling [4-6], and through resolved-sideband Raman cooling [7,8] in particular, it became possible to prepare an ion in the ground state of a harmonic trap. Thus, a broad range of elementary quantum physics was opened up to experimental study [9], including the spectroscopy [10-12] and photon statistics [13-15] of single-atom resonance fluorescence; quantized Rabi oscillation and engineered quantum states of motion [16-18] (through a “phonon” analogy of the Jaynes-Cummings model); and quantum decoherence, specifically, of superpositions of motional states [19,20]. Today, trapped ion systems continue to be developed as potential building blocks of quantum information processors [21-23].

Theoretical work on the quantized motion of trapped ions has mostly considered small displacements from equilibrium—the so-called Lamb-Dicke regime [24]. It is common for the ion to be manipulated in a standing-wave optical potential [25,26], in which case the potential is expanded to lowest order in the Lamb-Dicke parameter

$$\eta = \frac{2\pi}{\lambda} \sqrt{\frac{\hbar}{2m\omega_T}}, \quad (1)$$

where η is the ratio of the ion ground-state position uncertainty $\Delta x = \sqrt{\hbar/2m\omega_T}$ and the wavelength of the standing wave λ (multiplied by 2π); m is the ion mass and ω_T the frequency. With the restriction to the Lamb-Dicke regime, in many situations only one, two, or possibly a few quanta of excitation are considered [16,17,27,28].

The assumption $\eta \ll 1$ is a feature of the semiclassical treatment of atomic motion in laser light as well, where it underlies the diffusion or Fokker-Planck equation models [4,29-31]. These models assume that significant change to the particle (ion) momentum is built up from very many individual momentum “kicks” (each of size $\hbar k = h/\lambda$) with the wavepacket well-localized on the scale of λ ; thus, in a harmonic trap it is assumed that

$$\Delta p/\hbar k = (2k\Delta x)^{-1} = (2\eta)^{-1} \gg 1. \quad (2)$$

Such treatments also adiabatically eliminate the internal degrees of freedom, requiring the distance traveled in an atomic lifetime γ^{-1} to be much less than λ . This imposes the additional restriction

$$(\omega_T/\gamma)x_0 \ll \lambda, \quad (3)$$

where x_0 is the amplitude of oscillation in the trap. Since, reasonably, $x_0 > \Delta x = \eta\lambda/2\pi$, from Eqs. (2) and (3), the recoil energy is constrained by

$$\frac{\hbar^2 k^2/2m}{\hbar\gamma} = \eta^2 \omega_T/\gamma \ll 2\pi\eta. \quad (4)$$

In this paper we study the quantum motion of a trapped ion in laser light under conditions where none of the usual limitations and restrictions hold. Our aim is to exhibit the trapped ion system as an example of a *quantum stochastic process* in its fully elaborated form. Specifically, we consider an ion driven on a two-state electronic resonance by a standing-wave laser field—i.e., the heating of an ion through resonance fluorescence in a standing wave. Motion in one dimension is considered under the assumption of strong transverse confinement. The following features are notable in relation to the limitations and restrictions above:

- (i) heating rather than cooling of the ion takes place; thus, beginning in the ground state of the trap, the amplitude of oscillation grows to eventually extend over many periods of the standing wave, where energies (depending on η) as high as $10^4 \hbar \omega_T - 10^5 \hbar \omega_T$ are reached; thus, quantum motion well outside the Lamb-Dicke regime is considered;

- (ii) we focus upon the case

$$\omega_T/\gamma = 1, \quad (5)$$

so inequality (3) is violated at relatively small amplitudes of oscillation; a crucial interplay develops between the coherent evolution of the center-of-mass and internal degrees of freedom; even when a diffusion paradigm fits (for sufficiently small η), the motion is not understandable as diffusion in a prescribed harmonic plus optical potential;

- (iii) strong laser excitation, with Rabi frequency

$$\Omega/\gamma = 2, \quad (6)$$

is considered, but due to the interplay just noted the dressed-atom approach to atomic motion in laser light [32] cannot be used; modulation of the Rabi frequency by the harmonic center-of-mass motion plays a central role;

- (iv) Lamb-Dicke parameters ranging from $\eta = 0.2$ to $\eta = 3.0$ are considered, thus demonstrating the transition from a quasi-classical dynamic, for $\eta \ll 1$, to a manifestly *quantum* stochastic dynamic when $\eta > 1$; this transition is similar to that from weak- to strong-coupling in the treatment of quantum noise in cavity QED [33].

The reported investigation makes use of full *quantum trajectory* simulations [34], quantizing both the internal and the center-of-mass degrees of freedom. As an aid to understanding, approximate, *semi-quantum trajectory* simulations are explored as well, where the center-of-mass motion is treated classically apart from the inclusion of momentum “kicks” coordinated with the fluorescence. There have been previous quantum trajectory (Monte Carlo wave-function [35]) simulations of atomic motion in laser light [36–38], but all, to our knowledge, were carried out for conditions satisfying the restrictions of Eqs. (2)–(4).

We begin in Sec. 2 with an overview, presenting a summary of results in the form of heating curves (mean energy of the ion as a function of time) calculated for a series of Lamb-Dicke parameters ranging between $\eta = 0.2$ and $\eta = 3.0$. Results calculated from semi-quantum and quantum trajectories are compared and a number of differences identified for explanation in subsequent sections. The equations underlying the Monte-Carlo simulations are presented in Sec. 3, where we demonstrate a surprising feature of the individual realizations of the stochastic process; a stepwise evolution of the amplitude of oscillation of the ion is observed, pointing to the existence of a series of metastable amplitudes on which the ion heating almost stops. In Sec. 4 the origin of this evolution is identified as frequency modulation of the internal state Rabi oscillation. The metastable amplitudes of oscillation are identified with zeros of the J_0 Bessel function and characterized by the mean waiting time for fluorescent scattering plotted as a function of the amplitude of oscillation in the trap. A phase-space diffusion model is derived that recovers the heating curves of Sec. 2 in the limit of small Lamb-Dicke parameters, and quantitative differences between the semi-quantum and quantum diffusion are explained. Finally, in Sec. 5, Lamb-Dicke parameters of order unity are considered, where differences between a diffusion and a jump process are revealed. Here semi-quantum and quantum trajectory models are qualitatively different in their predictions. An observed quantum suppression of the ion heating rate is explained by the delocalization of the center-of-mass wavepacket across the standing wave.

2. Quantum and semi-quantum heating rates

We consider an ion of mass m trapped harmonically in one dimension with trap frequency ω_T . A classical laser field resonantly excites a closed two-state electronic transition of the ion, with decay rate γ and Rabi frequency Ω ; parameters chosen as in Eqs. (5) and (6). The laser field forms a standing wave along the axis of the ion motion (wavelength λ) and the equilibrium of the potential is located at an anti-node of the standing wave.

Figure 1 shows the growth of the mean-squared amplitude of oscillation of the ion—measured in wavelengths [see Eqs. (20) and (21)]—as a function of time for various Lamb-Dicke parameters η . Two separate models were used to compute these results as ensemble averages over Monte-Carlo simulations of the ion heating through fluorescence. For the results of Fig. 1(a), the ion center-of-mass position and momentum are treated as classical variables, but with a momentum “kick” added when a laser photon is scattered; the scattering is simulated as a quantum trajectory for a two-state system with a definite, though oscillating, position within the standing wave [34-37,39]. For the results of Fig. 1(b), full quantum trajectory simulations were carried out with both the center-of-mass and internal degrees of freedom quantized. The following points are of note and will be elaborated upon in the following sections:

- (i) for small Lamb-Dicke parameters both sets of results approach a limit where the heating curve becomes independent of η , apart from a factor η^4 that may be absorbed in the scaling of time; in this limit the heating is well-described by a diffusion in phase-space;
- (ii) semi-quantum and quantum trajectory models disagree quantitatively in the diffusion limit [compare the thinnest curves in Figs. 1(a) and (b)];
- (iii) larger Lamb-Dicke parameters yield an η -dependence over and above the scaling of time by η^4 ; thus, differences between a quantum jump and diffusion process become explicitly apparent;
- (iv) for large Lamb-Dicke parameters the two sets of curves are qualitatively different; the semi-quantum trajectories [frame (a)] show a heating rate that increases monotonically with increasing η , while the full quantum trajectory results [frame (b)] eventually show a dramatic reduction of the heating rate.

Before taking up these four themes, we first present the mathematical details of our two models in Sec. 3. Then we explore individual realizations of the quantum stochastic heating of the ion. These exhibit a very interesting evolution of the amplitude of oscillation of the ion in the trap, an underlying structure to the stochastic dynamics quite unanticipated on the basis of the ensemble averages displayed in Fig. 1.

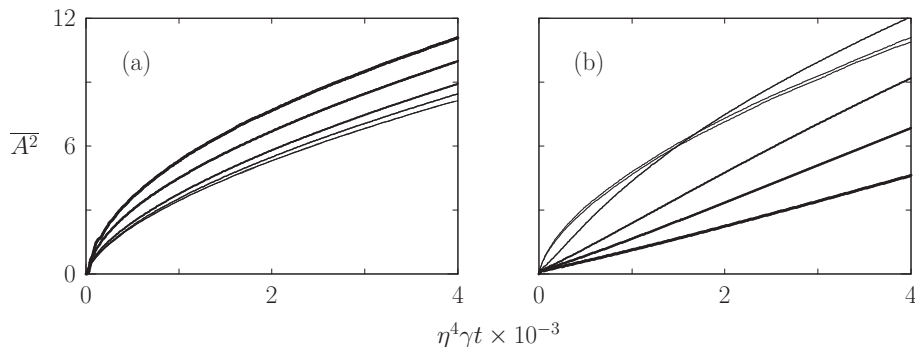


Fig. 1. Heating of a harmonically trapped ion through resonance fluorescence in a standing-wave field. Plots of the mean squared amplitude of oscillation as a function of time from: (a) semi-quantum trajectory simulations—Lamb-Dicke parameters $\eta = 0.2, 0.4, 0.8, 1.4, 2.2$, and 2.6 (thinnest line to thickest line); (b) quantum trajectory simulations—Lamb-Dicke parameters $\eta = 0.8, 1.4, 2.2, 2.6, 2.8$, and 3.0 (thinnest line to thickest line).

3. Monte-Carlo simulations

3.1 Semi-quantum trajectories:

Semi-quantum trajectories treat the center-of-mass motion of the ion classically. Only the electronic states of the ion are quantized. We represent the center-of-mass state by the point in phase-space

$$\alpha(t) = \tilde{\alpha}(t)e^{-i\omega_T t} = \sqrt{\frac{m\omega_T}{2\hbar}} \left[x(t) + i \frac{p(t)}{m\omega_T} \right], \quad (7)$$

where $x(t)$ denotes the oscillator position and $p(t)$ its momentum. The trajectory equations are formulated in a rotating frame, with $\tilde{\alpha}(t)$ denoting the complex amplitude of the oscillation. Following the standard quantum trajectory treatment of resonance fluorescence (see Sec. 8.2 of Ref. [34]), the conditional internal state of the ion is expanded in the interaction picture as

$$|\bar{\psi}_{\text{REC}}(t)\rangle = \bar{c}^{(-)}(t)|-\rangle + \bar{c}^{(+)}(t)|+\rangle, \quad (8)$$

where $|-\rangle$ and $|+\rangle$ are, respectively, the ground and excited electronic states, and the label REC refers to the conditioning of the state on the scattering record prior to time t ; ket vectors and Schrödinger amplitudes with an overbar, as in Eq. (8), are unnormalized. Under the assumed resonant excitation, in between photon scattering events, the (unnormalized) Schrödinger amplitudes obey the equations of motion

$$\begin{aligned} \frac{d\bar{c}^{(-)}}{dt} &= -(\Omega/2) \cos[\eta(\tilde{\alpha}e^{-i\omega_T t} + \tilde{\alpha}^*e^{i\omega_T t})] \bar{c}^{(+)}, \\ \frac{d\bar{c}^{(+)}}{dt} &= (\Omega/2) \cos[\eta(\tilde{\alpha}e^{-i\omega_T t} + \tilde{\alpha}^*e^{i\omega_T t})] \bar{c}^{(-)} - \frac{\gamma}{2} \bar{c}^{(+)}, \end{aligned} \quad (9)$$

where, importantly, the Rabi frequency Ω is modulated by the harmonic motion of the ion through the standing wave—using Eqs. (1) and (7), $\cos[kx(t)] = \cos\{\eta[\alpha(t) + \alpha^*(t)]\}$. Photon scattering is accounted for through quantum jumps, which interrupt this coherent evolution. The jumps occur at random times, with instantaneous rate

$$R_{\text{jump}}(t) = \gamma \frac{|\bar{c}^{(+)}(t)|^2}{|\bar{c}^{(-)}(t)|^2 + |\bar{c}^{(+)}(t)|^2}. \quad (10)$$

Each jump returns the ion to the ground electronic state,

$$\bar{c}_- \rightarrow \bar{c}_+, \quad \bar{c}_+ \rightarrow 0, \quad (11a)$$

and at the same time we add a momentum “kick” to the center-of-mass. The magnitude of this “kick” is determined by projecting the recoil for photon scattering in a randomly chosen direction (θ, ϕ) onto the line of the ion motion; thus, for photon scattering at time t , we implement a momentum “kick” in the rotating frame through the semi-quantum jump

$$\tilde{\alpha} \rightarrow \tilde{\alpha} + i\eta \sin \theta \cos \phi e^{i\omega_T t}, \quad (11b)$$

which follows from the magnitude of the recoil $\hbar k$ and Eqs. (1) and (7). The angle ϕ is uniformly distributed between 0 and 2π , while θ is distributed according to the dipole radiation pattern (oriented perpendicular to the ion motion).

A Monte-Carlo simulation must decide at each time step whether to propagate the internal state according to Eq. (9), leaving $\tilde{\alpha}$ unchanged, or to implement the quantum and semi-quantum jumps, Eqs. (11a) and (11b); the branching ratio is given by the jump probability $R_{\text{jump}}\Delta t$. When a jump (photon scattering) does occur, two random numbers determine the angles θ and ϕ . The heating of the ion results from the accumulated momentum “kicks” (11b). The mechanism is therefore, in principle, straightforward. It is important to note, however, that the “kick” rate is not a fixed function of either space or time, but itself evolves stochastically according to Eqs. (9) and (10). This gives rise to an unanticipated complexity in the ion motion, as illustrated in Sec. 3.4.

3.2 Quantum trajectories:

The full quantum trajectory treatment of the ion heating is formulated as a natural extension of Eqs. (8)–(11b). The principle difference is that Schrödinger amplitudes $\bar{c}^{(-)}(t)$ and $\bar{c}^{(+)}(t)$ are replaced by center-of-mass ket vectors, with the expansion of the total quantum state, quantized internal *and* center-of-mass degrees of freedom,

$$|\bar{\psi}_{\text{REC}}(t)\rangle = |\bar{\psi}_{\text{REC}}^{(-)}(t)\rangle|-\rangle + |\bar{\psi}_{\text{REC}}^{(+)}(t)\rangle|+\rangle. \quad (12)$$

The coupled equations of motion (9) for Schrödinger amplitudes are replaced by coupled equations of motion for the (unnormalized) center-of-mass kets,

$$\begin{aligned} \frac{d|\bar{\psi}_{\text{REC}}^{(-)}\rangle}{dt} &= -(\Omega/2) \cos[\eta(\hat{a}e^{-i\omega_T t} + \hat{a}^\dagger e^{i\omega_T t})] |\bar{\psi}_{\text{REC}}^{(+)}\rangle, \\ \frac{d|\bar{\psi}_{\text{REC}}^{(+)}\rangle}{dt} &= (\Omega/2) \cos[\eta(\hat{a}e^{-i\omega_T t} + \hat{a}^\dagger e^{i\omega_T t})] |\bar{\psi}_{\text{REC}}^{(-)}\rangle - \frac{\gamma}{2} |\bar{\psi}_{\text{REC}}^{(+)}\rangle, \end{aligned} \quad (13)$$

where \hat{a} and \hat{a}^\dagger are oscillator annihilation and creation operators, and replace the classical phase-space variables $\tilde{\alpha}$ and $\tilde{\alpha}^*$. In line with the rotating frame of reference used in Eqs. (7) and (9), the interaction picture of the harmonic oscillator has been adopted in Eqs. (12) and (13). The jump (photon scattering) rate is calculated from the center-of-mass ket norms in a straightforward generalization of Eq. (10),

$$R_{\text{jump}}(t) = \gamma \frac{\langle \bar{\psi}_{\text{REC}}^{(+)}(t) | \bar{\psi}_{\text{REC}}^{(+)}(t) \rangle}{\langle \bar{\psi}_{\text{REC}}^{(-)}(t) | \bar{\psi}_{\text{REC}}^{(-)}(t) \rangle + \langle \bar{\psi}_{\text{REC}}^{(+)}(t) | \bar{\psi}_{\text{REC}}^{(+)}(t) \rangle}, \quad (14)$$

and (11a) and (11b) are combined in the quantum jump

$$|\bar{\psi}_{\text{REC}}\rangle \rightarrow \hat{D}(i\eta \sin \theta \cos \phi e^{i\omega_T t}) |\bar{\psi}_{\text{REC}}^{(+)}|-\rangle, \quad (15)$$

where $\hat{D}(\xi) = \exp(\xi \hat{a}^\dagger - \xi^* \hat{a})$ is the displacement operator.

Monte-Carlo simulations may be carried out on the basis of Eqs. (13)–(15) in essentially the same manner as before. The numerical demands are significantly higher, though, due to the much larger number basis states involved. The two center-of-mass kets are expanded in the energy representation of the harmonic oscillator, and we have, in fact, carried out simulations where the energy of the center-of-mass motion rises to be of the order of $10^4 \hbar \omega_T - 10^5 \hbar \omega_T$. This is achieved by carrying out the calculations in a local frame that tracks the average complex amplitude of the oscillation, $\langle \hat{a}(t) \rangle_{\text{REC}}$, reached at the conclusion of a quantum jump. This strategy also circumvents the need to explicitly implement the displacement in Eq. (15).

3.3 Quantum trajectories in the local frame:

We absorb the mean displacement of the center-of-mass kets from one quantum jump to another into a change of the frame of reference. Thus, we work with the displaced center-of-mass kets,

$$|\bar{\psi}_{\text{local}}^{(\pm)}(t)\rangle = \hat{D}[-\tilde{\alpha}(t)] |\bar{\psi}_{\text{REC}}^{(\pm)}(t)\rangle, \quad (16)$$

where $\hat{D}[-\tilde{\alpha}(t)]$ displaces the center-of-mass state to a “local frame”. The local frame changes in time, discontinuously, at the times of the quantum jumps. When it has been decided that a photon scattering is to take place and the associated quantum jump (15) is to be executed, a displacement of the post-jump kets is made to first cancel the displacement of the momentum “kick” [Eq. (15)], and second to move the mean complex amplitude of the oscillator back to zero—i.e., to off-set any change in the mean brought about by the Schrödinger evolution during the interval that has elapsed since the last quantum jump. Then in place of Eqs. (13) we have equations of motion for the ket vectors in the local frame,

$$\begin{aligned} \frac{d|\bar{\psi}_{\text{local}}^{(-)}\rangle}{dt} &= -(\Omega/2) \cos\{\eta[(\tilde{\alpha} + \hat{a})e^{-i\omega_T t} + (\tilde{\alpha}^* + \hat{a}^\dagger)e^{i\omega_T t}]\} |\bar{\psi}_{\text{local}}^{(+)}\rangle, \\ \frac{d|\bar{\psi}_{\text{local}}^{(+)}\rangle}{dt} &= (\Omega/2) \cos\{\eta[(\tilde{\alpha} + \hat{a})e^{-i\omega_T t} + (\tilde{\alpha}^* + \hat{a}^\dagger)e^{i\omega_T t}]\} |\bar{\psi}_{\text{local}}^{(-)}\rangle - \frac{\gamma}{2} |\bar{\psi}_{\text{local}}^{(+)}\rangle, \end{aligned} \quad (17)$$

with jump rate

$$R_{\text{jump}}(t) = \gamma \frac{\langle \bar{\psi}_{\text{local}}^{(+)}(t) | \bar{\psi}_{\text{local}}^{(+)}(t) \rangle}{\langle \bar{\psi}_{\text{local}}^{(-)}(t) | \bar{\psi}_{\text{local}}^{(-)}(t) \rangle + \langle \bar{\psi}_{\text{localC}}^{(+)}(t) | \bar{\psi}_{\text{local}}^{(+)}(t) \rangle}. \quad (18)$$

The quantum jump is now executed by setting the mean oscillator amplitude with respect to the local frame to zero (and setting the ion in the ground electronic state), with

$$|\bar{\psi}_{\text{local}}^{(-)}\rangle \rightarrow \hat{D}(-\langle \hat{a} \rangle_{\text{local}}) |\bar{\psi}_{\text{local}}^{(+)}\rangle, \quad |\bar{\psi}_{\text{local}}^{(+)}\rangle \rightarrow 0, \quad (19a)$$

and moving the location of the local relative to the global frame, with

$$\tilde{\alpha} \rightarrow \tilde{\alpha} + \langle \hat{a} \rangle_{\text{local}} + i\eta \sin \theta \cos \phi e^{i\omega_T t}. \quad (19b)$$

In these expression $\langle \hat{a} \rangle_{\text{local}}$ is the conditional oscillator amplitude expectation calculated in the pre-jump local frame.

3.4 Sample results:

Figure 2 presents examples of the simulated trajectories for four different values of the Lamb-Dicke parameter, increasing from $\eta = 0.2$ in frame (a) to $\eta = 2.2$ in frame (d). We plot the amplitude of oscillation, measured in optical wavelengths, against time in atomic lifetimes scaled by η^4 (see Sec. 4.2). Explicitly, for semi-quantum trajectories, using Eqs. (1) and (7), the amplitude is given by

$$A = \frac{1}{\lambda} \sqrt{\frac{2\hbar}{m\omega_T}} |\tilde{\alpha}| = \frac{\eta}{\pi} |\tilde{\alpha}|, \quad (20)$$

while for quantum trajectories

$$A = \frac{\eta}{\pi} \sqrt{\langle \hat{a}^\dagger \hat{a} \rangle_{\text{REC}}} = \frac{\eta}{\pi} \sqrt{\langle (\tilde{\alpha}^* + \hat{a}^\dagger)(\tilde{\alpha} + \hat{a}) \rangle_{\text{local}}}. \quad (21)$$

There are two principal points of interest to be noted in the figure. First, frames (a) and (b) shown a clear step-like evolution, which is at least qualitatively similar in the semi-quantum and quantum simulations. Long periods of almost no heating at all (metastable amplitudes) are present, interspersed with periods of much more rapid heating, thus making up the steps. Quantitatively, periods of essentially no heating are observed to last as long as 10^6 lifetimes, while the metastable amplitudes are spaced approximately $\lambda/2$ apart—at $A = 0.38, 0.88, 1.38$, etc.. Second, the similarity between semi-quantum and quantum trajectories seems to be breaking down in frames (c) and (d). The quantum trajectory shows no clear steps in frame (d), while the stepping of the semi-quantum trajectory no longer keeps to the metastable amplitudes identified before. The task of the following sections is to explain these features along with points (i)–(iv) of Sec. 2.

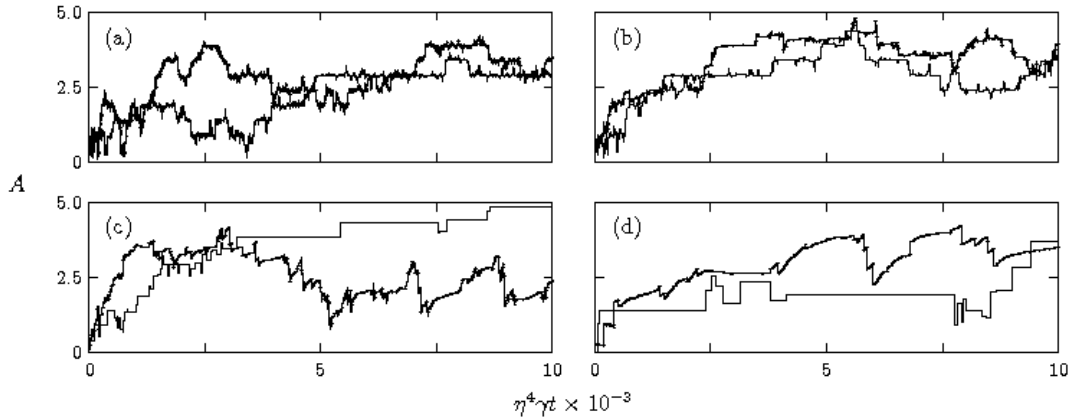


Fig. 2. Sample quantum (thick curves) and semi-quantum (thin curves) trajectory simulations of the amplitude of oscillation in the trap; for Lamb-Dicke parameters $\eta = 0.2$ (a), 0.8 (b), 1.4 (c), and 2.2 (d).

4. Mean waiting times and the diffusion limit

4.1 Semi-quantum waiting times:

An explanation of the step-like evolution of Figs. 2(a) and (b) is provided by Fig. 3. Here, in the upper series of pictures, we plot the induced Rabi oscillation and simulated photon scattering sequence for four different amplitudes of oscillation in the trap. For simplicity, only Semi-quantum trajectories are considered in this section. Frames (a) and (c) show the ion being fully excited, while (b) and (d) show much lower levels of excitation. The corresponding numbers of scattered photons are high—frames (a) and (c)—and much lower—frames (b) and (d). Evidently, the response of the ion to the modulated driving field [Eqs. (9)],

$$(\Omega/2) \cos[kx(t)] = (\Omega/2) \cos\{2\pi A \cos[\omega_T t - \arg(\tilde{\alpha})]\}, \quad (22)$$

depends strongly on the amplitude of oscillation in the trap. The observation is readily understood from the time-averaged Rabi frequency, $\Omega J_0(2\pi A)$, which is zero at the zeros of the J_0 Bessel function. The first two of these zeros occur at the amplitudes $A = 0.38$ and $A = 0.88$ of frames (b) and (d) in the figure, while amplitudes $A = 0.0$ and $A = 0.6$ [frames (a) and (c)] correspond to local maxima of the Bessel function. Thus, the metastable amplitudes are located by the zeros of $J_0(2\pi A)$, where the ion is only weakly excited and the fluorescence rate drops. Of course, some photon scattering occurs at these zeros, since the actual response is to the modulated driving and not to its time average; the amount decreases as A increases, as seen in the comparison between frames (b) and (d).

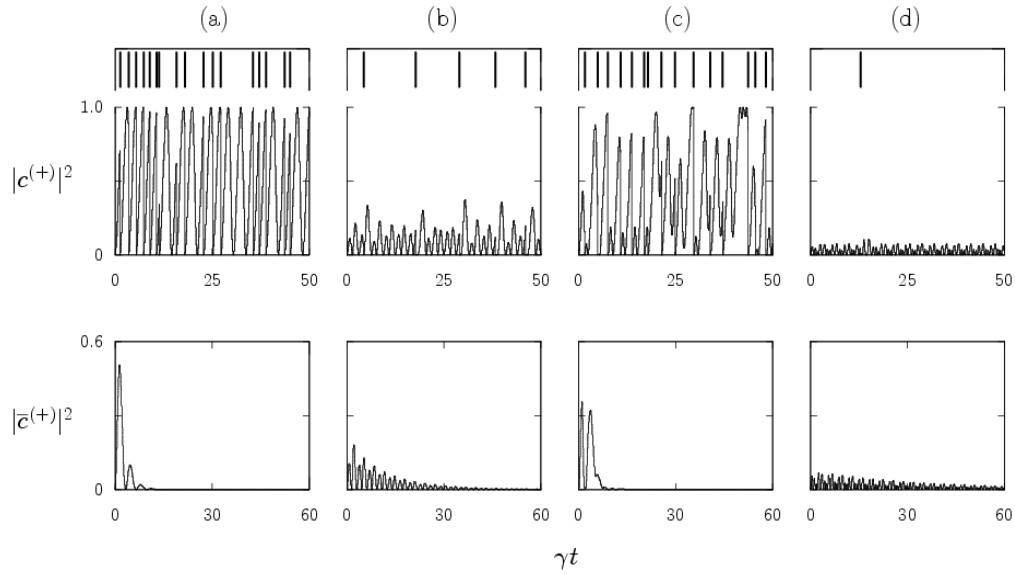


Fig. 3. Semi-quantum Rabi oscillation and photon scattering sequence (upper frames) and corresponding waiting-time distribution (lower frames) for four different amplitudes of oscillation in the trap: $A = 0.0$ (a), 0.38 (b), 0.6 (c), and 0.88 (d).

A useful way to quantify the behavior as a function of A is through the distribution of times waited between two successive scattering events (quantum jumps)—the so-called waiting-time distribution. This distribution is plotted for semi-quantum trajectories in the lower series of pictures in Fig. 3, where it is narrow (width of order $\gamma\tau = 1$) in frames (a) and (c), significantly broadened in frame (b), and broader still in frame (d). The distribution depends, most generally, on both the amplitude, A , and phase,

$$\zeta = \omega_T t - \arg(\tilde{\alpha}), \quad (23)$$

of the oscillation in the trap at the time of the quantum jump that begins the interval waited τ . The quantity plotted in Fig. 3 has been averaged over ζ . For given amplitude and phase, the waiting-time distribution is computed within quantum trajectory theory as [39]

$$W(\tau; A, \zeta) = \gamma |\bar{c}^{(+)}(t + \tau)|^2, \quad (24)$$

where $\bar{c}^{(+)}(t + \tau)$ satisfies Eqs. (9) with initial condition $\bar{c}^{(+)}(t) = 0$, $\bar{c}^{(-)}(t) = 1$. Its mean,

$$\bar{\tau}(A, \zeta) = \int_0^\infty d\tau \tau W(\tau; A, \zeta), \quad (25)$$

is sufficient to characterize the metastable states. For semi-quantum trajectories, an approximate analytical expression for this mean can be derived as follows.

The task is to solve Eqs. (9) with the initial internal state the electronic ground state. We first make a transformation to consider the complex variable

$$\bar{c}^{(-)}(t + \tau) + i\bar{c}^{(+)}(t + \tau) = B(\tau; A, \zeta)e^{i\beta(\tau; A, \zeta)}, \quad (26)$$

where $B(\tau; A, \zeta)$ and $\beta(\tau; A, \zeta)$ are real functions of τ . The equations of motion for Schrödinger amplitudes, Eqs. (9), then yield a solution in the form

$$B(\tau; A, \zeta) = \exp\left[-\frac{\gamma}{2} \int_0^\tau d\tau' \sin^2 \beta(\tau'; A, \zeta)\right], \quad (27)$$

with the phase $\beta(\tau; A, \zeta)$ satisfying the equation

$$\frac{d\beta(\tau; A, \zeta)}{d\tau} = -(\Omega/2) \cos[2\pi A \cos(\omega_T \tau + \zeta)] - \frac{\gamma}{4} \sin \beta(\tau; A, \zeta). \quad (28)$$

From Eqs. (24), (26), and (27), the waiting-time distribution is

$$W(\tau; A, \zeta) = -\frac{d}{d\tau} B^2(\tau; A, \zeta), \quad (29)$$

with mean waiting time, integrating Eq. (25) by parts,

$$\bar{\tau}(A, \zeta) = \int_0^\infty d\tau B^2(\tau; A, \zeta). \quad (30)$$

The development up to this point is exact. We now introduce an approximation which holds good whenever the mean waiting time is much larger than γ^{-1} . Assuming $\beta(\tau; A, \zeta)$ is very small, such that $\sin \beta(\tau; A, \zeta) \approx \beta(\tau; A, \zeta)$, Eq. (27) becomes

$$B(\tau; A, \zeta) = \exp\left[-\frac{\gamma}{2} \int_0^\tau d\tau' \beta^2(\tau'; A, \zeta)\right], \quad (31)$$

while Eq. (28) can be solved for

$$\beta(\tau; A, \zeta) = (\Omega/2) \text{Re} \left[\sum_{n=-\infty}^{\infty} (-i)^n \frac{J_n(2\pi A)}{\gamma/2 + in\omega_T} e^{in(\omega_T \tau + \zeta)} \right], \quad (32)$$

where we have used the standard Fourier series expansion of $\cos[2\pi A \cos(\omega_T \tau + \zeta)]$ [40]. We then replace $\beta^2(\tau; A, \zeta)$ in Eq. (31) by its d.c. component (which is independent of the phase ζ), and from Eqs. (29)–(31), the waiting-time distribution is approximated by the exponential distribution

$$W(\tau; A) = \bar{\tau}^{-1} e^{-\tau/\bar{\tau}(A)}, \quad (33)$$

with mean waiting time

$$\gamma \bar{\tau}(A) = \left[(\Omega/2)^2 \sum_{n=0}^{\infty} \frac{J_{2n}^2(2\pi A) + J_{-2n}^2(2\pi A)}{(\gamma/2)^2 + (2n\omega_T)^2} \right]^{-1}. \quad (34)$$

Note how the dominant term in the series expansion of $\gamma\bar{\tau}$ vanishes for the identified metastable amplitudes—when $J_0(2\pi A) = 0$. The approximation (34) is compared with the exact result computed from a semi-quantum trajectory average in Fig. 4. The two results are in remarkably good agreement everywhere except for amplitudes $A < 1$ and in a narrow range of amplitudes in between the peaks. We see that with increasing center-of-mass energy, the rate of photon scattering on the metastable amplitudes very quickly drops by more than two orders of magnitude. Clearly this interplay of the coherent oscillation of the ion in the trap and the induced Rabi oscillation must be the principal determinant of the heating curves—for small Lamb-Dicke parameters at least—and of the noted differences between semi-quantum and quantum models (Fig. 1).

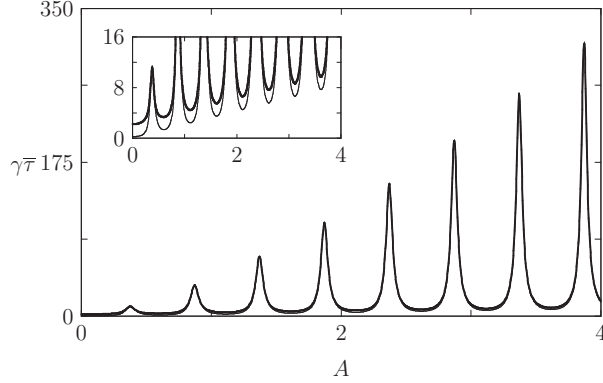


Fig. 4. Exact (thick curve) and approximate [Eq. (34)] (thin curve) semi-quantum mean waiting time plotted as a function of the amplitude of oscillation in the trap.

4.2 The semi-quantum diffusion limit:

Point (i) in Sec. 2 is concerned with the approach of the heating curves for small η to a diffusion limit. It was noted that the η -dependence in this limit amounts to a scaling of the time by η^4 . We are now in a position to elaborate upon these observations. We treat the diffusion limit for semi-quantum trajectories first, then compare the description for quantum trajectories in the following section.

It is convenient to use a set of coordinates rotated to remove the phase of the complex amplitude $\tilde{\alpha}$. In this frame of reference, the momentum “kick” defined by Eq. (11b) is written as [ζ defined by Eq. (23)]

$$\Delta(\theta, \phi, \zeta) = \Delta_{\parallel}(\theta, \phi, \zeta) + i\Delta_{\perp}(\theta, \phi, \zeta) = i\eta \sin \theta \cos \phi e^{i\zeta}, \quad (35)$$

where $\Delta_{\parallel}(\theta, \phi, \zeta)$ is a “kick” in the amplitude of $\tilde{\alpha}$ and $\Delta_{\perp}(\theta, \phi, \zeta)$ is a “kick” in its phase. Successive “kicks” are distributed randomly, though not necessarily uniformly, over the angles θ , ϕ , and ζ . Taking an average over these distributions yields a covariance matrix that depends only on the amplitude of oscillation in the trap A :

$$\mathbf{C}(A) = \mathbf{B}(A)\mathbf{B}^T(A) = \begin{pmatrix} \overline{\Delta_{\parallel}^2(\theta, \phi, \zeta)} & \overline{\Delta_{\parallel}(\theta, \phi, \zeta)\Delta_{\perp}(\theta, \phi, \zeta)} \\ \overline{\Delta_{\parallel}(\theta, \phi, \zeta)\Delta_{\perp}(\theta, \phi, \zeta)} & \overline{\Delta_{\perp}^2(\theta, \phi, \zeta)} \end{pmatrix}. \quad (36)$$

The covariance matrix defines the r.m.s. size of the phase-space “kicks”. It is plotted from a simulated trajectory average in Fig. 5. To this information we add their average rate—i.e., the inverse of the mean waiting time plotted in Fig. 4, also a function only of A . Assuming, then, that the phase-space “kicks”, of order η [Eq. (35)], are small, so that significant changes in $\tilde{\alpha}$ accumulate over very many photon scattering events, we may put these pieces of information together to arrive at the stochastic differential equation with amplitude-dependent diffusion

$$\mathbf{R}[-\arg(\tilde{\alpha})] \begin{pmatrix} d\tilde{\alpha}_{\parallel} \\ d\tilde{\alpha}_{\perp} \end{pmatrix} = \frac{1}{\sqrt{\bar{\tau}(A)}} \mathbf{B}(A) \begin{pmatrix} dW_{\parallel} \\ dW_{\perp} \end{pmatrix}, \quad (37)$$

where the rotation matrix $\mathbf{R}[-\arg(\tilde{\alpha})]$ brings the phase-space increments back to the original coordinate system; dW_{\parallel} and dW_{\perp} are independent Wiener increments, with covariances

$$\overline{dW_{\parallel}dW_{\parallel}} = \overline{dW_{\perp}dW_{\perp}} = dt, \quad \overline{dW_{\parallel}dW_{\perp}} = 0. \quad (38)$$

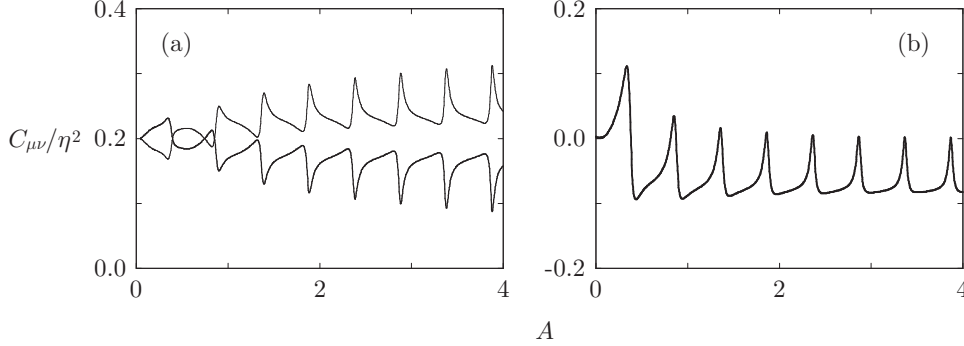


Fig. 5. Semi-quantum covariance matrix elements plotted as a function of the amplitude of oscillation in the trap: (a) amplitude (thick curve) and phase (thin curve) quadrature variances; (b) amplitude-phase cross-correlation.

The noted η^4 -scaling of the time axis in the diffusion limit [point (i) of Sec. 2] follows directly from Eq. (35)–(37). Note first that $\mathbf{B}(A) \sim \eta$, while, in addition, the oscillation amplitude in wavelengths is related to the phase-space amplitude by $A = \eta|\tilde{\alpha}|/\pi$ [Eq. (20)]; thus, after multiplying Eq. (37) by η/π , the right-hand side is of order η^2 and the fluctuation variance, at fixed A , grows as $\eta^4 t$.

We find that the diffusion model agrees with semi-quantum jump simulations for Lamb-Dicke parameters on the order of $\eta = 0.2$ or less; in fact, there is very little difference for $\eta = 0.4$ [results for $\eta = 0.2$ and 0.4 are indistinguishable in Fig. 1(a)]. Further increase, however, brings noticeably faster semi-quantum heating, as shown by the curves for $\eta = 0.8, 1.4, 2.2$, and 2.6 in Fig. 1(a) [point (iii) of Sec. 2]. In this regime the scale of the individual quantum events (momentum “kicks”) approaches that of the underlying phase-space structure represented by the peaks in Fig. 4. A handful of jumps can destabilize the metastable amplitudes much faster than the diffusion process can.

4.3 Quantum trajectories compared:

Semi-quantum trajectories are conceptually simple and provide a convenient starting point for our understanding. They are nevertheless a rather gross approximation. Points (ii) and (iv) of Sec. 2 record two instances where the approximation has a significant effect upon the heating rate. Considering first the quantitative difference in the diffusion limit [point (ii)], we might ask how the mean waiting time and covariance matrix change as functions of the amplitude of oscillation A . The mean waiting time computed from full quantum trajectories is plotted in Fig. 6. For a Lamb-Dicke parameter of $\eta = 0.1$ it is almost indistinguishable from the semi-quantum result. At larger values of η , however, differences set in: there is a gradual smoothing out of the peaks, which is noticeable for $\eta = 0.2$ and virtually complete for $\eta = 1.4$. This development is explained in Sec. 5.

Changes to the covariance matrix are more relevant for the diffusion (small η) limit. Here the most important point to note is the difference between Eqs. (11a) and (19a). The former specifies the semi-quantum phase-space “kick”, while the latter adds to this the local-frame displacement of the center-of-mass wavepacket, $\langle \hat{a} \rangle_{\text{local}}$, which occurs during the elapsed waiting time following the last quantum jump. We might refer to this as the *stochastic dipole-force displacement*. It arises from the direct action of the standing-wave light field on the center-of-mass of the ion. Since resonant excitation constitutes a nonperturbative interaction, it is not possible to extract an optical potential to account for this. Nevertheless, to add to the momentum “kicks” from fluorescence, there is also momentum exchanged between the standing-wave laser field and the ion; this the semi-quantum model omits, while quantum trajectories include it as a time evolution of the center-of-mass kets— $|\bar{\psi}_{\text{REC}}^{(-)}(t)\rangle$ and $|\bar{\psi}_{\text{REC}}^{(+)}(t)\rangle$ —taking place in between the quantum jumps [Eqs. (17)].

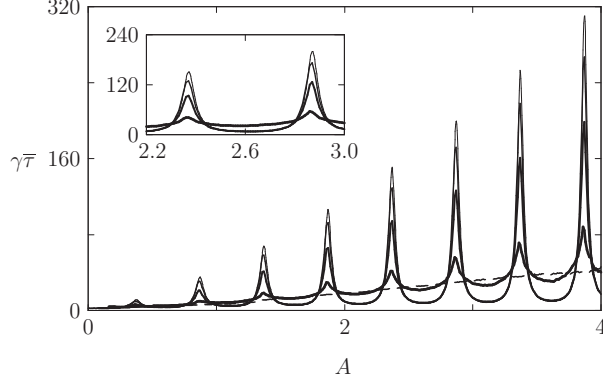


Fig. 6. Mean waiting time from quantum trajectories plotted as a function of the amplitude of oscillation in the trap; for Lamb-Dicke parameters $\eta = 0.2, 0.4, 0.8$ (thinner line to thicker line), and 1.4 (dashed line). The tallest curve (thinnest line) is the exact semi-quantum result plotted in Fig. 4.

From Eq. (19b), when characterizing the individual quantum events for quantum trajectories, we replace Eq. (35) by

$$\Delta(\theta, \phi, \zeta, \zeta') = |\langle \hat{a} \rangle_{\text{local}}| e^{i\zeta'} + i\eta \sin \theta \cos \phi e^{i\zeta}, \quad (39)$$

where a dependence enters upon the additional relative phase

$$\zeta' = \arg(\langle \hat{a} \rangle_{\text{local}}) - \arg(\tilde{\alpha}). \quad (40)$$

The covariance matrix is defined as before, with real and imaginary components of $\Delta(\theta, \phi, \zeta, \zeta')$ resolved in amplitude and phase directions, but now an average over the four phases $\theta, \phi, \eta, \eta'$ is taken. The numerically computed result is displayed in Fig. 7. Its principal difference when compared to Fig. 5 is that the phase variance peaks on the metastable amplitudes to reach a value an order of magnitude larger than in the semi-quantum case. Other less obvious changes are observed; for example, the amplitude variance also increases to produce a broad background near each metastable amplitude—the likely cause of the noted larger heating rate [point (ii) of Sec. 2].

The peaking of the phase variance on the metastable amplitudes appears most dramatically when phase-space plots are made. Sample plots are shown in Fig. 8, where we plot the complex amplitude $\tilde{\alpha}$ from semi-quantum simulations (second column in the figure) and $\langle \hat{a} \rangle_{\text{REC}}$ from quantum simulations (third column). In the diffusion limit ($\eta = 0.2$), the phase diffusion on the metastable amplitudes produces a dramatic “bulls-eye” pattern for quantum trajectories (top right frame), in sharp contrast

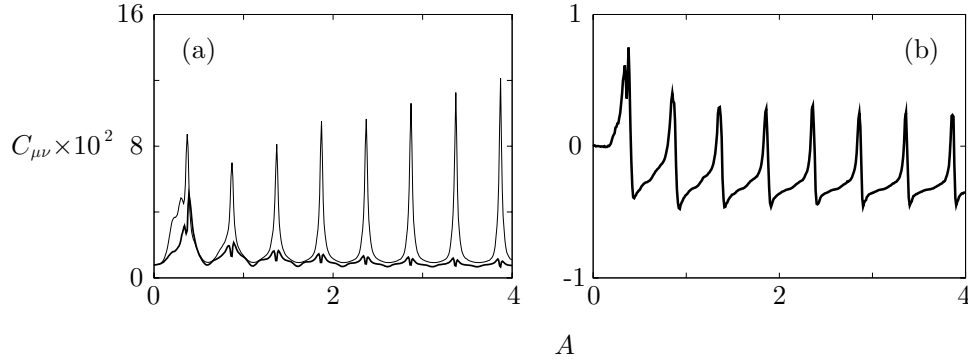


Fig. 7. Full quantum covariance matrix elements plotted as a function of the amplitude of oscillation in the trap: (a) amplitude (thick curve) and phase (thin curve) quadrature variances; (b) amplitude-phase cross-correlation. Results computed as a Monte-Carlo average for $\eta = 0.2$; note that a small sampling error remains.

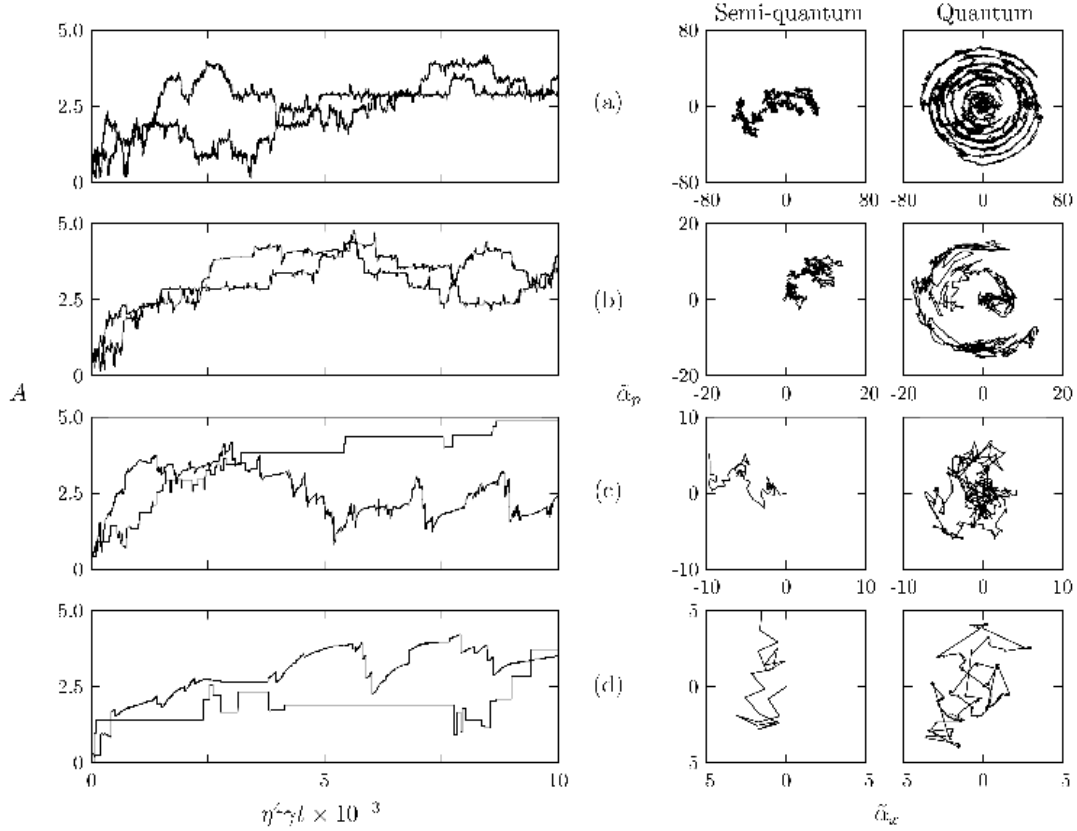


Fig. 8. Sample semi-quantum (second column) and quantum (third column) trajectory simulations compared through plots of the complex oscillator amplitude in phase space; for Lamb-Dicke parameters $\eta = 0.2$ (a), 0.8 (b), 1.4 (c), and 2.2 (d). The first column reproduces the plots from Fig. 2.

to the semi-quantum result. The phase diffusion is still in evidence for $\eta = 0.8$ (second row), but gradually ceases to be a feature as η is further increased and the diffusion picture breaks down.

Our final comment on the comparison between semi-quantum and quantum trajectories for small Lamb-Dicke parameters is illustrated by Fig. 9. Quantum trajectories provide a quantum description of the center-of-mass state—they evolve a center-of-mass wavepacket. What, then, is the form of this state? Perhaps we can expect a good approximation to a minimum uncertainty state, which is the closest quantum mechanical representation of the phase-space point of Eq. (7). As it turns out, this is not the case, even for the smallest Lamb-Dicke parameters considered (though for sufficiently small η we might still expect that result). The states shown as Q functions in Fig. 9 were recovered for a Lamb-Dicke parameter $\eta = 0.2$. They illustrate a generally observed amplitude squeezing, with large squeezing in the vicinity of the metastable amplitudes and significantly less squeezing at amplitudes in between. The degree of squeezing—of the conditional (instantaneously sampled) wavepacket—is plotted as a function of the amplitude of oscillation in the trap in the frame to the right in the figure.

We propose the following quantum measurement interpretation of these results. The ion fluorescence may be viewed as the record of an imperfect measurement, one revealing information about the amplitude of oscillation in the trap. Most simply, if the fluorescence rate is low, the ion likely oscillates with one of the metastable amplitudes; if high, with some amplitude midway in between. Beyond this simplest statement, a more quantitative deduction about the actual amplitude of oscillation may be made on the basis of the correlation between fluorescence rate and oscillation amplitude given in Figs. 4 and 6. The important point to note is that the amplitude discrimination achieved is especially high in the vicinity of a metastable amplitude and significantly poorer in between. Where the discrimination is high, the center-of-mass wavepacket becomes squeezed so that its predicted uncertainty in the amplitude of oscillation is consistent with the data made available through the measurement record (fluorescence)—i.e. the squeezing

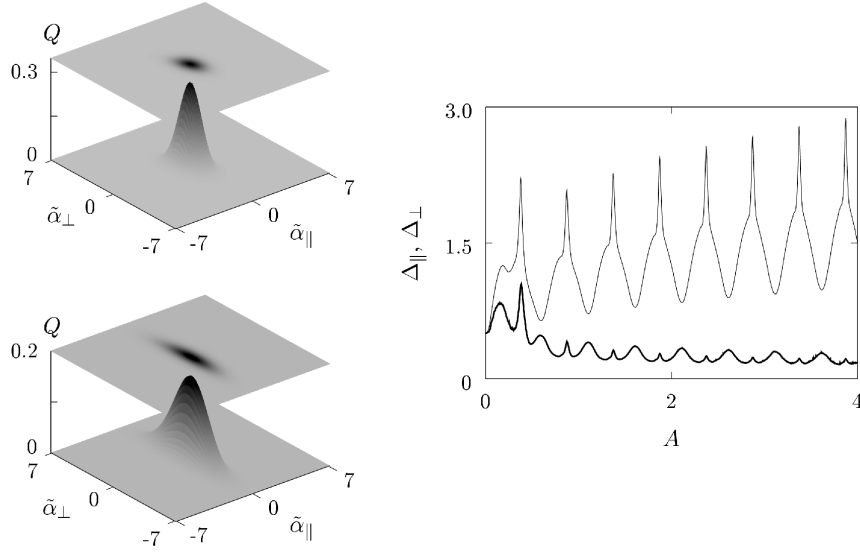


Fig. 9. Squeezing of the center-of-mass wavepacket as a function of the amplitude of oscillation in the trap. The scattered photons may be detected and read as the signal of an imperfect measurement of the amplitude A . Near a metastable amplitude, the discrimination of the measurement is very good and the center-of-mass wavepacket becomes squeezed so as to be consistent with the record read (Q function on the lower left); in between the metastable amplitudes the discrimination, consequently the squeezing, is not so good (Q function on the upper left). To the right, amplitude (thick curve) and phase (thin curve) quadrature variances are plotted as a function of the amplitude of oscillation in the trap. All results are for a Lamb-Dicke parameter $\eta = 0.2$.

may be view as the product of a dynamical “collapse of the wavepacket”, which in the case of a near-ideal measurement would produce an amplitude eigenstate.

5. Quantum inhibition of fluorescence and the limit of large quantum jumps

Points (iii) and (iv) of Sec. 2 note that large Lamb-Dicke parameters yield an η -dependence in the heating curves of Fig. 1 over and above the η^4 -scaling of the time axis—the dependence arising from the diffusion limit. Point (iv) notes, in particular, that the quantum heating rate *decreases* for large η , while the increase seen for semi-quantum trajectories seems more reasonable, when, due to their increased size, just a few momentum “kicks” are sufficient to “skip over” (destabilize) the metastable amplitudes. In fact, the lowered heating rate is caused by a suppression of fluorescence of an entirely different (to the mechanism of Sec. 4) quantum mechanical origin, as we now explain.

Consider, for example, the matrix elements for excitation of the ion out of the ground electronic state and the ground state of the trap while absorbing one photon and an even number of “phonons”; the final state is the excited electronic state and energy eigenstate state of the trap $|2n\rangle$, $n = 0, 1, 2, \dots$. The matrix elements, as a function of η , are

$$\begin{aligned} \langle 2n | \cos[k\hat{x}(t)] | 0 \rangle &= \langle 2n | \cos \left[\eta \left(\hat{a} e^{-i\omega_T t} + \hat{a}^\dagger e^{i\omega_T t} \right) \right] | 0 \rangle \\ &= (-1)^n \frac{\eta^{2n}}{\sqrt{2n!}} e^{-\eta^2/2} e^{i2n\omega_T t}. \end{aligned} \quad (41)$$

Note that matrix elements for the absorption of an odd number of “phonons” vanish due to the even parity of $\cos(k\hat{x})$. The squares of matrix elements (41) are plotted for the first few n in Fig. 10(a). From the figure we see that transitions $|0\rangle \rightarrow |0\rangle$ dominate in the diffusion limit. On the other hand, for $\eta \sim 1$, the ground-state wavepacket extends beyond $\pm\lambda/4$ where the cosine function changes sign. With the sign change the value of the matrix element is reduced. Thus, at $\eta = 1.5$, the $|0\rangle \rightarrow |2\rangle$ transition is strongest, while for $\eta \sim 2$, $|0\rangle \rightarrow |4\rangle$ and $|0\rangle \rightarrow |6\rangle$ are the strongest transitions. We note then that these strongest,

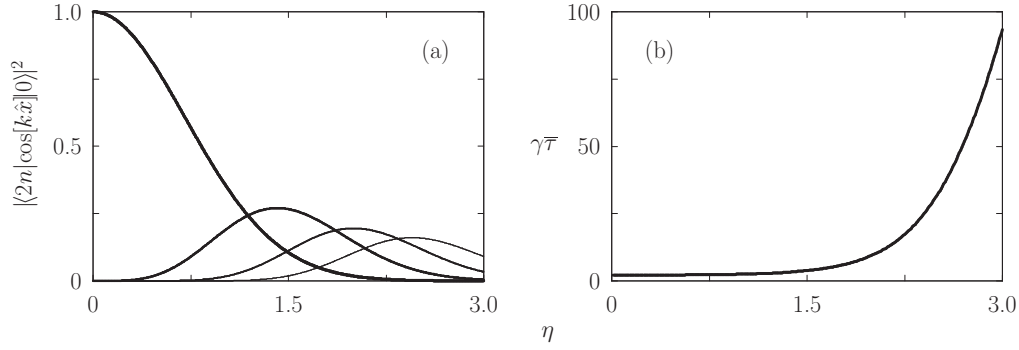


Fig. 10. (a) Matrix elements for transitions out of the ground state of the trap, $|0\rangle$, to the excited state $|2n\rangle$, plotted as a function of Lamb-Dicke parameter η for $n=0, 1, 2$, and 3 (thicker curve to thinner curve); (b) Mean waiting time until the first quantum jump as a function η .

multi-phonon transitions are not excited on resonance; the $|0\rangle \rightarrow |2n\rangle$ transition is detuned by an amount $2n\omega_T = 2n\gamma$ [Eq. (5)]. The combined effect of these two observations leads to a strong quantum suppression of the fluorescence; hence the reduced heating rate of Fig. 1(b). As an illustration of the suppression, in Fig. 10(b) we plot, as a function of η , the mean waiting time—beginning in the ground state of the trap—for the very first photon to be scattered. The waiting time grows rapidly for Lamb-Dicke parameters larger than $\eta = 1.5$.

There is a great deal more to be said about the quantum-stochastic motion of a trapped ion in the regime of large Lamb-Dicke parameters. This regime is similar to that of optical frequency cavity QED, where the influence of single scattering events is no longer small and the diffusion picture fails, or becomes forced at best [32]. As an indication of what is on offer, we conclude with Fig. 11. Here we present

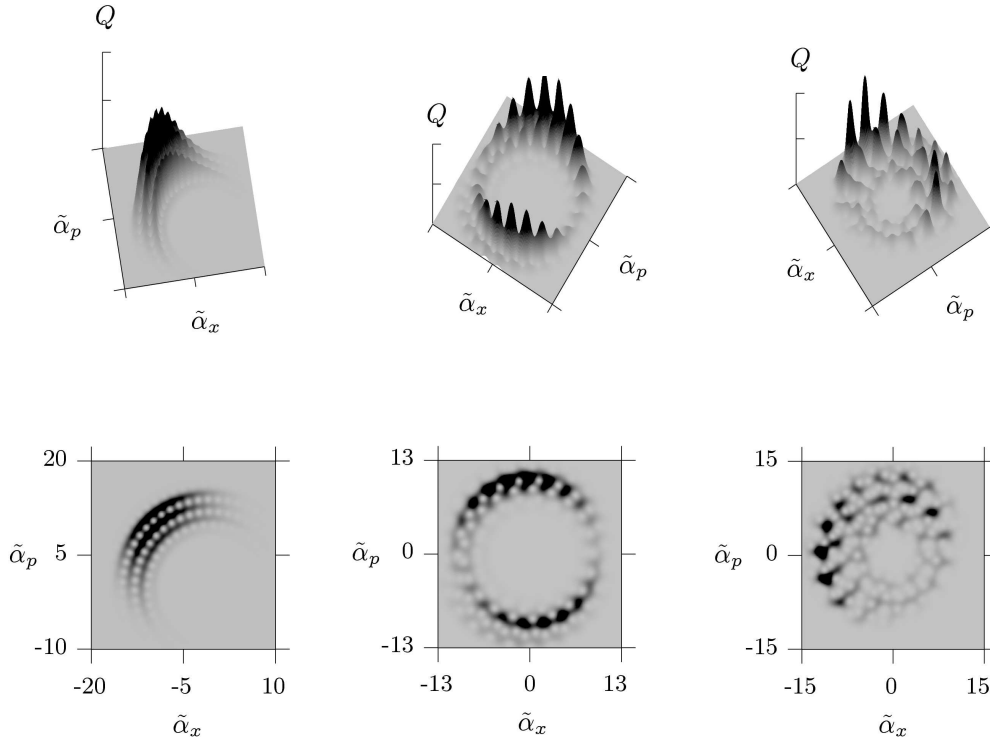


Fig. 11. Sample states from quantum trajectories for Lamb-Dicke parameters $\eta=0.8$ (left plots), 1.4 (center plots), and 2.2 (right plots). Q functions for the conditional center-of-mass state are plotted.

Q functions for sample states reached with Lamb-Dicke parameters $\eta = 0.8, 1.4$, and 2.2 . Of course, a huge variety of such states are visited throughout the course of a quantum trajectory; the question of what interesting quantities might be easily measured has yet to be explored. Note, however, the progression from the amplitude squeezed state of Fig. 9 to the states with increasing phase uncertainty of Fig. 11. Note also the interesting development of the amplitude uncertainty. The state for $\eta = 0.8$ (on the left in the figure), for example, resides on three metastable amplitudes at once, as a superposition. Then there is the development from left to right—with increasing η —where, for $\eta = 2.2$, a somewhat disordered phase-space structure emerges from the quantum interference pattern. Clearly, there remains much interesting behavior to be investigated in future work.

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